

Critical Schwinger pair production II - universality in the deeply critical regime

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We study electron-positron pair production by spatially inhomogeneous electric fields. Depending on the localization of the field, a critical point (critical surface) exists in the space of field configurations where the pair production probability vanishes. Near criticality, pair production exhibits universal properties similar to those of continuous phase transitions. We extend results previously obtained in the semi-classical (weak-field) critical regime to the deeply critical regime for arbitrary peak field strength. In this regime, we find an enhanced universality, featuring a unique critical exponent $\beta = 3$ for all sufficiently localized fields. For a large class of field profiles, we also compute the non-universal amplitudes.

I. INTRODUCTION

Universality is a paradigm that often arises from the dominance of long-range fluctuations near critical points, washing out the effect of microscopic details on the long-range observables. This form of universality can be cast into scaling laws of observables which are characterized by universal critical exponents that depend only on a few gross features of the system such as dimensionality, symmetries and the number and nature of the long-range degrees of freedom. Standard examples are provided by critical phenomena in spin systems or liquid-gas transitions [1–3].

Beyond fluctuation dominated systems, universality has also become a useful concept in classical (deterministic) systems such as turbulence [4], or even general relativity [5, 6], where the resulting scaling laws reflect self-similarity of the field configurations induced by the nonlinearities of the underlying theory over a wide range of scales.

In a recent letter [7], we have found aspects of universality also in Schwinger pair production [8–10], where an analogue of a critical point exists in the form of field configurations that provide the minimum of electrostatic energy to produce a (real) pair from vacuum, see e.g. [11–14]. On the one hand, this form of universality appears to fit into the framework of fluctuation-driven criticality, as the onset of pair production arises from long-range electron-positron quantum fluctuations that acquire sufficient energy from the external field to become real. On the other hand, the relevant physics of this process can be extracted from the Klein-Gordon or Dirac equation in an external field, which may be viewed as a classic deterministic and even linear wave equation.

This makes universality in Schwinger pair production a rather special example. Nevertheless, the origin of this universality has a clear physical picture: the relevant

long-range fluctuations average over the local details of the pair-producing field profile, giving rise to a scaling law that depends only on the large-scale properties of the field. We emphasize that this universality holds only near criticality. By contrast, the microscopic details of the field can play an important role in other parameter regions, such as in the dynamically assisted pair production regime [15–25]. The diversity of this phenomenon of pair production and its interpretation as a decay of the vacuum have lead to a search and study of analogue systems in e.g. atomic ionization [26], graphene [27, 28], and semiconductors [29, 30] and with ultracold atoms in optical lattices [31–34].

In our previous work [7], we interpreted the pair production probability $\text{Im } \Gamma$ as an order parameter and determined the scaling of this order parameter with the distance from the critical point. Using semiclassical worldline instanton methods [35–38], we found that the *semi-classical critical regime* entails a family of critical exponents β which is directly related to the power by which the electric field vanishes, i.e. the power d for asymptotically vanishing fields $E \sim x^{-d}$, or n for fields with compact support $E \sim (x - x_0)^n$. For each d and n there is one universality class. Though the worldline instanton approach facilitates a direct understanding of criticality and universality, and provides quantitative information about the semiclassical region near the critical point, the critical point itself actually lies outside the semiclassical regime of validity. This makes the large degree of universality that we found even more remarkable. It is natural to expect that universality will be enhanced in the immediate vicinity of the critical point. In this paper, we verify this expectation in the affirmative.

This paper is organized as follows. In Sect. II we give a general introduction as well as a brief summary of the main results. In Sect. III we derive the universal critical scaling of the probability for fields vanishing asymptotically faster than $|x|^{-3}$, and in Sect. IV and Sect. V we derive also the non-universal coefficient under the additional assumption of either strong or weak field strengths. In Sect. VI and Sect. VII we study fields that decay as $|x|^{-3}$ and $|x|^{-2}$, respectively. In Sect. VIII, we briefly

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consider spinor QED. We conclude in Sect. IX.

II. UNIVERSALITY IN SCHWINGER PAIR PRODUCTION

Schwinger pair production denotes the instability of the asymptotic “in” vacuum towards the creation of pairs in the presence of an external electric or electromagnetic field. The decay probability P of the vacuum is related to the imaginary part of the QED effective action Γ ,

$$P = 1 - \exp(-2 \operatorname{Im} \Gamma[E]) . \quad (1)$$

To lowest order, $\operatorname{Im} \Gamma$ hence is a measure for the pair production rate [39–41]. From the viewpoint of critical phenomena, we consider $\operatorname{Im} \Gamma$ as an order parameter for pair production. In the space of all conceivable electromagnetic field configurations, $\operatorname{Im} \Gamma$ can only be nonzero, if the external background can transfer sufficient energy to the electron-positron fluctuations to form a real pair. In the infinite dimensional space of field strength tensor functions, the regions where $\operatorname{Im} \Gamma \neq 0$ are therefore separated from those where $\operatorname{Im} \Gamma = 0$ by a critical hypersurface.

In the present work, we confine ourselves to a large class of field configurations within which we can approach the critical surface from the unstable-vacuum side ($\operatorname{Im} \Gamma \neq 0$) by tuning one parameter. For this, we consider unidirectional spatially inhomogeneous electric background fields with one nonzero vector component $E(x)$, which varies along the direction x of the field. For convenience, we use units with $\hbar = c = 1$ and absorb a factor of the electron charge into the background field, $eE \rightarrow E$. We concentrate on pair production to leading-order, ignoring radiative corrections of the photon field which would involve higher orders in the fine-structure constant $\alpha = e^2/(4\pi)$, see, e.g. [42–52]. We also use units in which the rest mass of the electron is set to $m = 1$, implying that all dimensionful quantities are expressed in units of the electron mass.

The fields of interest can be parametrized by $E = A'$ with potential

$$A(x) = \frac{1}{\gamma}(1 + f(kx)), \quad \gamma = \frac{k}{E_0}, \quad (2)$$

where E_0 is a characteristic field strength scale and k^{-1} a characteristic length scale of the inhomogeneous field. Their precise choice is not relevant. In fact, the field profile may support various of these scales, such that the function $f(kx)$ in addition depends on dimensionless ratios of further scales. Of particular relevance is the adiabaticity parameter $\gamma = k/E_0$, as we limit ourselves to fields with

$$f(-\infty) < f(kx) < f(\infty) \quad (3)$$

and normalize the profile function f such that $f(\pm\infty) = \pm 1$. As a consequence, $A(-\infty) = 0$ and $A(+\infty) = 2/\gamma$.

This class of fields goes beyond those considered frequently in the literature, in particular there is no restriction concerning monotonicity and (anti-)symmetry of f .

A semi-classical viewpoint suggests that pair production requires the electric field be sufficiently strong or extended to provide an electrostatic energy greater than the energy of a real pair at rest. In the full quantum theory, this threshold may receive quantum radiative corrections from final-state interactions, which come, however, with higher powers of α . Recalling that $m = 1$, the energy constraint reads

$$\int_{-\infty}^{\infty} dx E > 2 \implies \gamma < 1 . \quad (4)$$

In our previous paper [7], we have studied criticality in the semiclassical regime,

$$E_0^2 \ll 1 - \gamma^2 \ll 1 . \quad (5)$$

We have considered fields that decay asymptotically with a power law $E \rightarrow E_0 c(kx)^{-d}$ or vanish at a finite point x_0 as $E \rightarrow E_0 c(k[x - x_0])^n$. In the semiclassical regime, we have [7, 37]

$$\operatorname{Im} \Gamma \sim \frac{\exp[-\frac{\pi}{E}g(\gamma^2)]}{(\gamma^2 g)' \sqrt{(\gamma^2 g)''}} \quad (\dots)' := \frac{d}{d\gamma^2}(\dots) \quad (6)$$

and, for $n > 1$ and $d > 3$, the critical limit is obtained from

$$g = \frac{2}{\pi} \int_{-\infty}^{\infty} du \sqrt{1 - f^2(u)} + C(1 - \gamma^2)^\rho + \dots , \quad (7)$$

where $C(s, c)$ and $\rho = \frac{1}{2} \frac{s+3}{s+1}$, with $s = n, -d$, only depend on the asymptotic behavior of the field. The scaling for $n \leq 1$ and $d \leq 3$ can be obtained from second term in Eq. (7) and its first and second derivative, see [7]. Differentiating the second term in Eq. (7) gives terms that cause the prefactor of $\operatorname{Im} \Gamma$ to vanish as $\gamma \rightarrow 1$. Although $\rho > 0$ for $d > 3$, the second term can also be quantitatively relevant in the exponent for $1 - \gamma^2 > E_0^2$. For $\rho < 0$ we find essential scaling. Thus, the scaling is determined by the second term in Eq. (7), and hence only depends on the asymptotic behavior of the field. So, the power d or n groups fields into different universality classes in the semi-classical regime.

The semi-classical critical regime defined in Eq. (5) is likely to be most relevant to upcoming experiments, as the field strength or intensity rather than length scales represent the most challenging issue, e.g. for high-power lasers. Still, limiting the criticality study to this regime is conceptually not satisfactory, as the criticality limit, i.e., the approach of the critical surface is defined by taking $\gamma \rightarrow 1$, say for constant E_0 . It is the aim of the present work to study scaling in the *deeply critical regime*, defined by the regime where $1 - \gamma^2$ is smaller than any other scale,

$$1 - \gamma^2 \ll \{E^2, 1/E^2, 1\} . \quad (8)$$

Below, we show that the scaling in this regime depends even less on the field. In fact, for all n and $d > 3$ we find power-law scaling with the same universal critical exponent,

$$\text{Im } \Gamma \propto (1 - \gamma^2)^3. \quad (9)$$

If we further assume that the field is weak, $E^2 \ll 1$, we find a general expression for the non-universal coefficient

$$\text{Im } \Gamma = P(n, d)(1 - \gamma^2)^3 e^{-2S}, \quad (10)$$

where the prefactor P is universal in the sense that it depends only on the asymptotic behavior of the electric field (see below), and the microscopic details of the field are included in the “tunneling exponent”

$$S = \int_{-\infty}^{\infty} du \frac{\sqrt{1 - f^2}}{k}, \quad u = kx, \quad (11)$$

which we recognize from the semiclassical result Eq. (7). The result (10) holds for different combinations of d and n , e.g. fields decaying with d_+ for $x \rightarrow \infty$ and d_- for $x \rightarrow -\infty$, or decaying with d for $x \rightarrow \infty$ and with n at a finite point. By taking the limit $d \rightarrow \infty$ or $n \rightarrow \infty$ we recover the scaling for an exponentially decaying field, which agrees in particular with the exact result for the Sauter field [39]. We will see below, that the scaling differs from $(1 - \gamma^2)^3$ for weak fields vanishing slower than $|x|^{-3}$.

Importantly, the critical scaling $(1 - \gamma^2)^3$ also holds without assuming $E \ll 1$. Many aspects of criticality of Schwinger pair-production are independent of the spin of the created particles; hence, it suffices to perform the study for the simpler case of scalar QED for most aspects. An interesting difference between scalar and spinor QED occurs though for strong fields $E^2 \gg 1$. For scalar QED, we find for $E^2 \gg 1$ that the final result for the order parameter is given by (recall $f^2 \leq 1$)

$$\text{Im } \Gamma_{\text{scal}} = \frac{L^2 T}{48\pi} E_0^2 (1 - \gamma^2)^3 \left(\int_{-\infty}^{\infty} du 1 - f^2(u) \right)^{-2}. \quad (12)$$

In fact, (12) holds also for fields which vanish slower than $|x|^{-3}$ as long as the integral in (12) converges. Strong fields vanishing as $|x|^{-2}$ still have to be treated separately.

Spinor QED also exhibits the same scaling with $(1 - \gamma^2)^3$. Moreover, for strong fields, we find a remarkably universal expression

$$\text{Im } \Gamma_{\text{spin}} = \frac{L^2 T}{96\pi} (1 - \gamma^2)^3. \quad (13)$$

A comparison of Eq. (13) and Eq. (12) reveals two important differences: First, in contrast to the weak-field regime where scalar and spinor QED predict essentially the same $\text{Im } \Gamma$, here we find $\text{Im } \Gamma_{\text{spin}} \ll \text{Im } \Gamma_{\text{scal}}$ due the

factor of E_0^2 in Eq. (12). Second, spinor QED leads to a higher degree of universality; in fact, the whole expression Eq. (13) is universal and does neither depend on the details of the profile nor on the asymptotic behavior of the field.

Note that there is no exponential suppression factor in Eq. (12) or Eq. (13), as would be typical for pair production in weak fields (e.g. as in (10)). The large-field regime therefore appears most promising for a future experimental verification of this universal critical behavior. In scalar QED, the factor of $E_0^2 \gg 1$ in Eq. (12) can further compensate for the critical factor $(1 - \gamma^2)^3 \ll 1$, which could be important for analog systems.

III. DERIVATION OF THE UNIVERSAL CRITICAL EXPONENT

In the present work, we use the classical field equation for an analysis of the pair production probability, see [39, 40, 53, 54] for more details on this formalism. As the critical scaling is independent of spin, we will focus on the scalar case and solve directly the Klein-Gordon equation. Throughout we consider $1 - \gamma^2$ to be the smallest parameter in the problem and we work to the lowest nontrivial order. As the fields only depend on x , the Klein-Gordon equation can be written

$$(\partial_x^2 + [p_0 - A(x)]^2 - m_\perp^2) \varphi = 0, \quad (14)$$

where $m_\perp^2 = 1 + p_\perp^2$ and p_\perp is the momentum spatially transverse to the field direction. The momentum longitudinal to the field for $x \rightarrow -\infty$ is

$$p^2 := p_0^2 - m_\perp^2, \quad (15)$$

and becomes for $x \rightarrow \infty$

$$q^2 := (p_0 - 2/\gamma)^2 - m_\perp^2. \quad (16)$$

Following [39, 40, 53, 55], we are looking for the solution of (14) that behaves asymptotically as

$$J e^{ipx} + R e^{-ipx} \xleftarrow{x \rightarrow -\infty} \varphi(x) \xrightarrow{x \rightarrow \infty} e^{iqx}. \quad (17)$$

The imaginary part of the effective action is obtained [39, 53] by integrating the tunneling factor

$$\mathcal{T} = \frac{q}{p} \frac{1}{|J|^2} \quad (18)$$

over momentum,

$$\text{Im } \Gamma = \frac{L^2 T}{2} \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{dp_0}{2\pi} \mathcal{T}. \quad (19)$$

We are in the so-called Klein region, where the energy is in the classical tunnel regime, and the integration limits are obtained from (c.f. (15) and (16))

$$m_\perp < p_0 < \frac{2}{\gamma} - m_\perp. \quad (20)$$

We change variables in order to make the dependence on $1 - \gamma^2$ manifest,

$$p_{\perp}^2 = (1 - \gamma^2)r \quad p_0 = 1 + (1 - \gamma^2)\frac{v}{2}. \quad (21)$$

Working in an expansion in $1 - \gamma^2$, the asymptotic longitudinal momenta read to leading order

$$p^2 = (1 - \gamma^2)(v - r) \quad q^2 = (1 - \gamma^2)(2 - v - r), \quad (22)$$

with corrections being of order $\mathcal{O}[(1 - \gamma^2)^2]$. With this change of variables of Eq. (21) and working to leading order in $(1 - \gamma^2)$, the effective action becomes

$$\text{Im } \Gamma = \frac{L^2 T}{8(2\pi)^2} (1 - \gamma^2)^2 \int_0^1 dr \int_r^{2-r} dv \mathcal{T}. \quad (23)$$

We can already see that $\text{Im } \Gamma$ vanish at least as fast as $(1 - \gamma^2)^2$, in contrast to the semiclassical exponent that follows from Eq. (7). We will show that \mathcal{T} is linear in $1 - \gamma^2$ for a large class of fields, so that $\text{Im } \Gamma \sim (1 - \gamma^2)^3$.

For this, we begin by expanding the Klein-Gordon equation (14) to first order in $1 - \gamma^2$,

$$\left(\partial_u^2 - \frac{1 - f^2}{k^2} + \frac{1 - \gamma^2}{k^2} [f^2 + (1 - v)f - r] \right) \varphi = 0, \quad (24)$$

where $u = kx$. In order to find J and R in Eq. (17), we start with the asymptotic wave $\varphi = e^{iqu/k}$ for $u \sim k/q \gg 1$ and work backwards to $u \rightarrow -\infty$. For definiteness we introduce a bookkeeping parameter $\lambda \ll 1$ and define the asymptotic regions as

$$1 - f^2 \leq \lambda(1 - \gamma^2). \quad (25)$$

For the class of fields defined in Eq. (3), there are only two regions satisfying Eq. (25), which we refer to as the right ($f > 0$) and the left ($f < 0$) asymptotic region. In the right asymptotic region, the Klein-Gordon equation is simply

$$\left(\partial_u^2 + \frac{q^2}{k^2} \right) \varphi = 0, \quad (26)$$

where we have used Eq. (22) and $f \simeq 1 + \mathcal{O}(\lambda(1 - \gamma^2))$. According to Eq. (17), the solution is $\varphi = e^{iqu/k}$. Let us call the location of the border of the right asymptotic region u_{λ}^+ defined by $1 - f^2(u_{\lambda}^+) = \lambda(1 - \gamma^2)$ and $f(u_{\lambda}^+) > 0$. We claim that φ is a very slowly oscillating wave near u_{λ}^+ , so that we can choose an irrelevant phase such that we have $\varphi = 1$ at and near the border of the right asymptotic region. This claim follows trivially for fields that are identically zero for u larger than some u_0 , i.e. for fields with $f(u_0) = 1$ at $|u_0| < \infty$. For fields vanishing asymptotically, u_{λ}^+ is large so one has to be more careful to make sure that qu_{λ}^+/k is small. Using the defining equation for u_{λ}^+ and assuming that the field

decays asymptotically as $E \rightarrow E_0 c(kx)^{-d}$ with c being a dimensionless constant, we have

$$\frac{qu_{\lambda}^+}{k} = \frac{1}{k} \left(\frac{2c}{d-1} \frac{1}{\lambda} \right)^{\frac{1}{d-1}} q^{\frac{d-3}{d-1}}. \quad (27)$$

For fields decaying sufficiently fast, i.e., for $d > 3$, the plane wave phase (27) is indeed small, so $\varphi = e^{iqu/k} \approx 1$ for $u \sim u_{\lambda}^+$. For $d < 3$, on the other hand, (27) is large. Below, we see that weak fields with $d \leq 3$ and strong fields with $d = 2$ exhibit different scalings from $(1 - \gamma^2)^3$.

As u decreases from the right asymptotic region we first come to what we will refer to as the right semi-asymptotic region, which is delineated by $f > 0$ and

$$\lambda(1 - \gamma^2) \leq 1 - f^2 \leq \Lambda(1 - \gamma^2) \ll 1, \quad (28)$$

where $\Lambda \gg 1$ is another bookkeeping parameter. For reference, we indicate the right semi-asymptotic region by $u_{\Lambda}^+ < u < u_{\lambda}^+$. In this region, the second and the third term in (24) can be of the same order. Since we can still substitute $f = 1$ in the third term, the Klein-Gordon equation (24) reduces to

$$\left(\partial_u^2 - \frac{1 - f^2}{k^2} + \frac{q^2}{k^2} \right) \varphi = 0. \quad (29)$$

However, since both the second and third term in Eq. (29) are small, we can solve for φ perturbatively, which to lowest order is simply $\varphi = 1$.

In the left semi-asymptotic region, which satisfies Eq. (28) but with $f < 0$, the Klein-Gordon equation is given by replacing q^2 with p^2 in Eq. (29). For reference we indicate this region by $u_{\lambda}^- < u < u_{\Lambda}^-$. For fields vanishing in this region either at a finite point or asymptotically with $d > 3$, the solution to lowest order is given by $\varphi = C + Du$, where C and D are constants obtained by solving the Klein-Gordon equation in between the semi-asymptotic regions where $1 - f^2 \gg 1 - \gamma^2$. These constants depend on the microscopic details of the field in the region where the field strength is comparatively strong, but they are independent of $1 - \gamma^2$, because $\varphi = 1$ in the right semi-asymptotic region. Since $p|u|/k \ll 1$ in the left semi-asymptotic region we can write

$$\begin{aligned} \varphi &= C + Du \\ &\simeq \frac{kD}{2ip} \left\{ \exp \left[\frac{ip}{k} \left(u + \frac{C}{D} \right) \right] - \exp \left[-\frac{ip}{k} \left(u + \frac{C}{D} \right) \right] \right\}. \end{aligned} \quad (30)$$

By matching the semi-asymptotic form (30) with the asymptotic form (17) we find

$$|J| = \frac{k|D|}{2p}. \quad (31)$$

Note that $|R| = |J|$ holds to lowest order as expected, since $|J| \gg 1$. With Eq. (18) and Eq. (22) the tunneling factor becomes

$$\mathcal{T} = \frac{4pq}{k^2|D|^2} = \frac{4(1 - \gamma^2)}{k^2|D|^2} \sqrt{v - r} \sqrt{2 - v - r}. \quad (32)$$

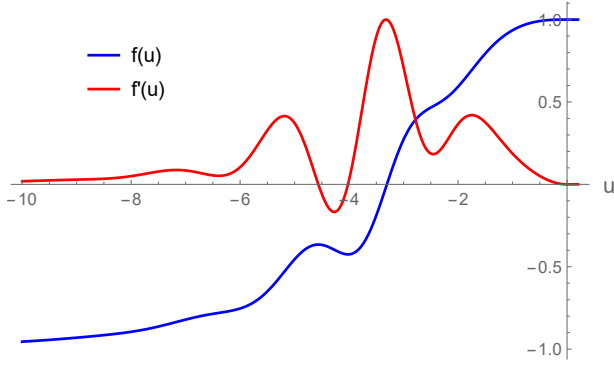


FIG. 1. Example field as defined in Eq. (123). The blue curve depicts the potential function $f(kx)$ and the red curve is the electric field $E(kx)/E_0 = f'$. The electric field has been chosen identically zero for $kx > 0$ and vanishes asymptotically as $|kx|^{-6}$ for $kx \rightarrow -\infty$.

After substituting this into Eq. (23) and performing the momentum integrals (v and r) we finally find

$$\text{Im } \Gamma = \frac{\pi L^2 T}{12} \frac{(1 - \gamma^2)^3}{E_0^2 |2\pi D|^2}. \quad (33)$$

Since D is independent of $1 - \gamma^2$ we see that all fields that vanish faster than $|x|^{-3}$ have the same power-law scaling $\text{Im } \Gamma \sim (1 - \gamma^2)^\beta$ with critical exponent $\beta = 3$. This scaling can be confirmed by comparing with the exact result [39] for a Sauter field. In order to arrive at Eq. (33), there was no need to assume $E \ll 1$ (or equivalently $k \ll 1$). In fact, in the next sections we will derive explicit expressions for the non-universal constant D in terms of e.g. d and n , both for weak fields $E \ll 1$ and for strong fields $E \gg 1$.

As a verification of Eq. (33), we have numerically solved the Klein-Gordon equation in its original form (14). The result for the field in Fig. 1 is shown in Fig. 2 and Fig. 3. We summarize the parametrizations of the field profiles used as illustrations in the appendix.

IV. D FOR STRONG FIELDS

Let us now derive the non-universal constant D , starting with the simpler case for strong fields $E_0 \gg 1$, the weak field case $E_0 \ll 1$ is treated in the next section.

For this, we need to solve the equation

$$\left(\partial_u^2 - \frac{1 - f^2}{k^2} \right) \varphi = 0 \quad (34)$$

in between the two semi-asymptotic regions, i.e., for $u_\Lambda^- < u < u_\Lambda^+$, and with boundary condition $\varphi = 1$ for $u \sim u_\Lambda^+$. Schematically, we need to find D in the solution chain

$$J e^{ipx} + R e^{-ipx} \xrightarrow{u_\Lambda^-} C + D u \xrightarrow{u_\Lambda^-} ? \xrightarrow{u_\Lambda^+} 1 \xrightarrow{u_\Lambda^+} e^{iqx}. \quad (35)$$

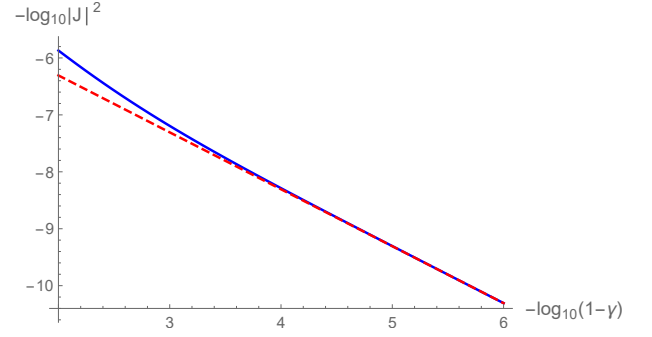


FIG. 2. Double-logarithmic plot of the inverse of the squared amplitude, $1/|J|^2$, as a function of γ for the field shown in Fig. 1. The blue curve is obtained by solving Eq. (14) numerically, and the red dashed line is a straight line with slope -1 obtained by matching with the blue curve at the end of the plot. The field strength is $E_0 = 1$ and the momentum parameters at $r = 0$ and $v = 1$. This plot demonstrates that sufficiently close to the critical surface $\gamma = 1$ the tunneling factor is linear in $1 - \gamma^2$ and contributes as such to $\text{Im } \Gamma$, which implies that $\text{Im } \Gamma \propto (1 - \gamma^2)^3$.

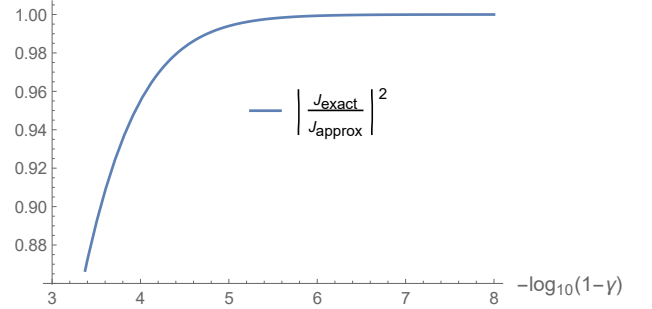


FIG. 3. Amplitude ratio for the field as in Fig. 2 with the same parameters. J_{exact} is the exact amplitude obtained from the numerical solution of Eq. (14), and $J_{\text{approx}} = (E_0/2p)\varphi'(u \rightarrow -\infty)$ is obtained from the numerical solution of Eq. (34) with boundary condition $\varphi = 1$ and $\varphi' = 0$ at $u = 0$. For $\gamma = 1 - 10^{-8}$ we have $|J_{\text{exact}}/J_{\text{approx}}|^2 \approx 1 - 7 \cdot 10^{-6}$.

The solution to Eq. (34) is obtained by expanding in $1/k^2$, which to lowest order gives

$$\varphi = 1 + \int_{u_\Lambda^-}^u du' (u - u') \frac{1 - f^2(u')}{k^2}, \quad u_\Lambda^- < u < u_\Lambda^+. \quad (36)$$

By taking $u \sim u_\Lambda^-$ we find

$$D = \int_{-\infty}^{\infty} du \frac{1 - f^2(u)}{k^2}, \quad (37)$$

where the extension of the integration boundaries remains exact to leading order where $f \simeq 1$ in all (semi-)asymptotic regions. Substituting Eq. (37) into Eq. (33) gives us Eq. (12).

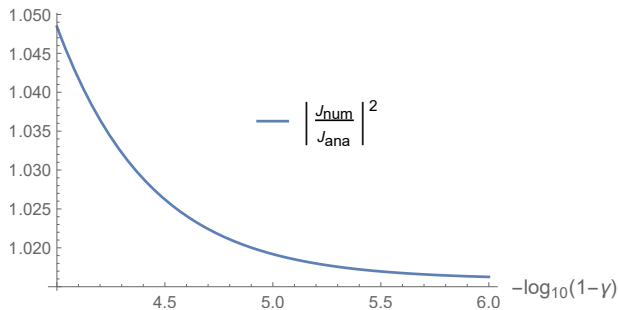


FIG. 4. Ratio of the squared amplitude $|J|^2$ obtained numerically and that obtained analytically from Eq. (37), as a function of $-\log_{10}(1-\gamma)$. The field shape is that shown in Fig. 1, the field strength is $E_0 = 30$, and the momentum parameters are $r = 0$ and $v = 1$. The plot shows that the analytical approximation for the tunneling factor $\mathcal{T} \propto 1/|J|^2$ is only slightly larger than the numerical result, and that the analytical approximation improves with $\gamma \rightarrow 1$. The asymptotic error can be made smaller by choosing a stronger field (recall that Eq. (37) has been derived under the assumptions $1 - \gamma^2 \ll 1/E^2 \ll 1$).

As a simple check of Eq. (37), consider a Sauter pulse, $f(u) = \tanh u$. From Eq. (37) and Eq. (32) it follows that $\mathcal{T} = qp k^2$, which is in perfect agreement with the exact solution [39, 53] in the regime $1 - \gamma^2 \ll 1/E^2 \ll 1$. As a further check, consider the field depicted in Fig. 1. In Fig. 4 we show that Eq. (37) agrees well with a numerical solution of the Klein-Gordon equation (14). Another field example is shown in Fig. 5, and the agreement between Eq. (37) and numerical results is demonstrated in Fig. 6. In Fig. 7 we demonstrate with a field that vanishes as $|x|^{-5/2}$ that Eq. (37) also holds for fields vanishing slower than $|x|^{-3}$ (compare though with the scaling found in Sect. VII for fields vanishing as $|x|^{-2}$). In each of these cases, the asymptotic error for $\gamma \rightarrow 1$, i.e., the slight deviations of the numerical to analytical amplitude ratio from the deeply critical value $|J_{\text{num}}/J_{\text{ana}}|^2 \rightarrow 1$, is controlled by the field strength. The deeply critical amplitude ratio is approached more closely, the better the parameter hierarchy $1 - \gamma^2 \ll 1/E^2 \ll 1$ of the deeply critical regime is satisfied. We emphasize that this asymptotic error does not affect the scaling property.

V. D FOR WEAK FIELDS

Let us now derive the non-universal coefficient D in Eq. (33) for weak fields $E_0 \ll 1$. We will consider fields that vanish either asymptotically as x^{-d} or beyond specific points x_0 as $(x - x_0)^n$; there are four different combinations.

We again have to solve Eq. (34) in the region between the two semi-asymptotic regions, $u_{\Lambda}^- < u < u_{\Lambda}^+$, and with boundary condition $\varphi = 1$ for $u \sim u_{\Lambda}^+$. This time we divide this region into three regions. We begin with the

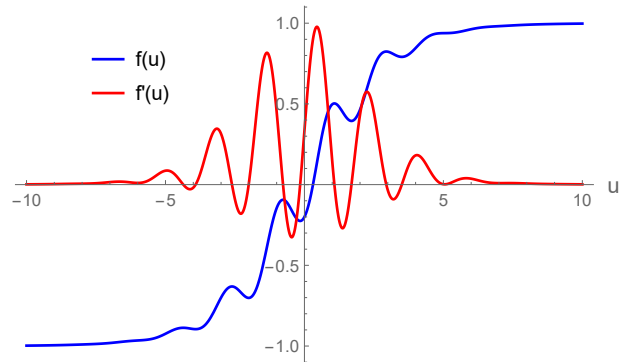


FIG. 5. Example field as defined in Eq. (124). The blue curve depicts the potential function $f(kx)$ and the red curve is the electric field $E(kx)/E_0 = f'$. The field decays exponentially for $|u| \rightarrow \infty$.

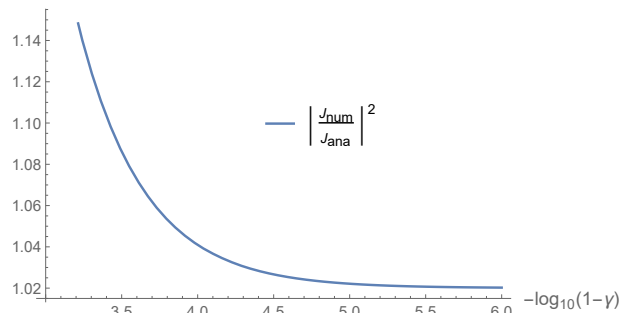


FIG. 6. Ratio of the squared amplitude $|J|^2$ obtained numerically and that obtained analytically from Eq. (37), as a function of $-\log_{10}(1-\gamma)$. The field shape is that shown in Fig. 5, the field strength is $E_0 = 30$, and the momentum parameters are $r = 0$ and $v = 1$. The plot shows that the analytical approximation improves with $\gamma \rightarrow 1$.

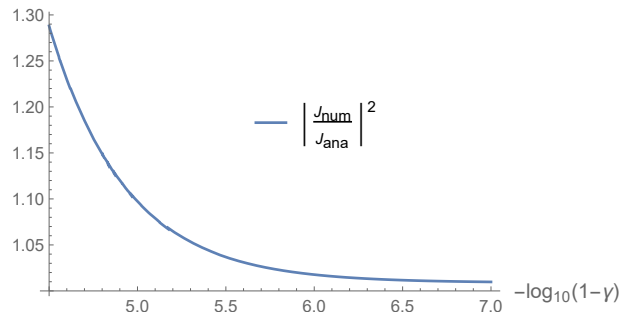


FIG. 7. Ratio of the squared amplitude $|J|^2$ obtained numerically and that obtained analytically from Eq. (37), as a function of $-\log_{10}(1-\gamma)$. The field shape is $f = u(1 + |u|^{3/2})^{-2/3}$, with field-strength parameter $E_0 = 200$, and momentum parameters $r = 0$ and $v = 1$. The plot shows that the strong-field approximation (12) is valid also for fields vanishing slower than $|x|^{-3}$. However, here we need larger E_0 for Eq. (37) to be a satisfactory approximation. This serves as an indication for the fact that we find a completely different scaling for weak fields vanishing slower than $|x|^{-3}$.

two outer regions where

$$1 - f^2 \leq Lk^2 \ll 1 \quad L \gg 1, \quad (38)$$

where L is yet another bookkeeping parameter. In these two regions we expand f around ± 1 . The lowest non-trivial order can be solved analytically in terms of Bessel functions. Therefore, we refer to these two regions as the left and right Bessel regions from now on. Defining the inner boundaries of these regions in terms of u_L^\pm by $1 - f(u_L^\pm)^2 = Lk^2$, where the \pm sign holds for $f > 0$ and $f < 0$, respectively, the left Bessel region covers $u_\Lambda^- < u < u_L^-$ and the right Bessel region $u_L^+ < u < u_\Lambda^+$.

Between the two Bessel regions, i.e. $u_L^- < u < u_L^+$, we have

$$\frac{1 - f^2}{k^2} \geq L \gg 1, \quad (39)$$

which implies that Eq. (34) can be well approximated by the standard WKB method,

$$\varphi = \frac{A}{(1 - f^2)^{\frac{1}{4}}} \exp \left(\int_u^\infty \frac{\sqrt{1 - f^2}}{k} \right). \quad (40)$$

We have dropped the term with opposite sign in front of the integral since it is exponentially smaller. The upper limit in the integral is chosen for convenience. This choice is possible for all n and for $d > 3$, but for $d \leq 3$ we have to choose a finite value.

To connect the WKB and the Bessel regions we note that near the boundaries $u \sim u_L^\pm$ we can express the solution in terms of either Bessel functions or as in Eq. (40). In other words, the region where the WKB form is valid partly overlap with the region where we can expand f around ± 1 . By expanding the Bessel functions in u near the boundary between the left Bessel region and the left semi-asymptotic region, i.e. at $u \sim u_\Lambda^-$, we obtain D in Eq. (30), which then completes the final result Eq. (33).

In summary, we wish to construct the solution chain for weak fields,

$$J e^{ipx} + R e^{-ipx} \xleftarrow{u_\Lambda^-} C + D u \xleftarrow{u_\Lambda^-} \text{Bessel} \xleftarrow{u_L^-} \text{WKB} \xleftarrow{u_L^+} \text{Bessel} \xleftarrow{u_\Lambda^+} 1 \xleftarrow{u_\Lambda^+} e^{iqx}. \quad (41)$$

A. The right Bessel region

1. Fields decaying asymptotically

We begin with the right Bessel region, $u_L^+ < u < u_\Lambda^+$, and with fields decaying as

$$E \rightarrow E_0 \frac{c_+}{(kx)^{d_+}} \quad x \rightarrow \infty, \quad (42)$$

where $d_+ > 3$. The $+$ subscripts indicate that we are in the right Bessel region; to avoid cumbersome notation we

will simply write d and c where the meaning should be clear from the context. The potential is

$$f = 1 - \frac{c}{d-1} u^{-(d-1)}, \quad (43)$$

and the Klein-Gordon equation reduces to

$$\left(\partial_u^2 - \frac{2c}{(d-1)k^2} u^{-(d-1)} \right) \varphi = 0. \quad (44)$$

The solution can be written in terms of Bessel functions [56]. The correct linear combination is determined by requiring that $\varphi \rightarrow 1$ as $u \rightarrow \infty$ (c.f. Eq. (41)). One obtains

$$\varphi = b_1 \sqrt{u} I \left(\frac{1}{d-3}, \frac{2}{d-3} \sqrt{\frac{2c}{d-1}} \frac{1}{k u^{\frac{d-3}{2}}} \right), \quad (45)$$

with normalization constant

$$b_1 = \Gamma \left(1 + \frac{1}{d-3} \right) \left(k(d-3) \sqrt{\frac{d-1}{2c}} \right)^{\frac{1}{d-3}}. \quad (46)$$

By matching Eq. (45) and the WKB form (40) for $u \sim u_L^+$, we find [56]

$$A = \sqrt{\frac{k(d-3)}{4\pi}} b_1. \quad (47)$$

2. Fields vanishing beyond a finite point

Before we connect with the left Bessel region we consider fields vanishing beyond a finite point u_+ as

$$E \rightarrow E_0 c_+ (u_+ - u)^{n_+}. \quad (48)$$

The potential is given by (again omitting $+$ subscripts for brevity)

$$f = 1 - \frac{c}{n+1} (u_+ - u)^{n+1}, \quad \text{for } u \lesssim u_+, \quad (49)$$

and $f = 1$ for $u \geq u_+$. The Klein-Gordon equation reduces to

$$\left(\partial_{v_+}^2 - \frac{2c}{(n+1)k^2} v_+^{n+1} \right) \varphi = 0, \quad (50)$$

where $v_+ = u_+ - u$. The general solution is again given by Bessel functions. Demanding that the solution and its derivative be continuous at u_+ , we find

$$\varphi = b_1 \sqrt{v_+} I \left(-\frac{1}{3+n}, \frac{2}{3+n} \sqrt{\frac{2c}{1+n}} \frac{v_+^{\frac{3+n}{2}}}{k} \right), \quad (51)$$

where

$$b_1 = \Gamma \left(1 - \frac{1}{3+n} \right) \left(\frac{1}{3+n} \sqrt{\frac{2c}{1+n}} \frac{1}{k} \right)^{\frac{1}{3+n}}. \quad (52)$$

By matching Eq. (51) with the WKB form (40) in the overlap region $u \sim u_L^+$, we find [56]

$$A = \sqrt{\frac{k(3+n)}{4\pi}} b_1. \quad (53)$$

Note that the similarity between the asymptotic (d) case in Eq. (47) and the compact (n) case here. To highlight this, we let s stand for either n or $-d$, and define

$$F(s, c) := |3+s|^{\frac{1}{2}} \Gamma\left(1 - \frac{1}{3+s}\right) \left(\frac{1}{|3+s|} \sqrt{\frac{2c}{|1+s|}} \frac{1}{k}\right)^{\frac{1}{3+s}}. \quad (54)$$

This allows us to write the WKB constant in Eq. (40) as

$$A(s, c) = \sqrt{\frac{k}{4\pi}} F(s, c). \quad (55)$$

B. The left Bessel region

Next we perform a similar matching with the Bessel region to the left of the WKB region at u_L^- .

1. Fields vanishing asymptotically

We begin again with fields decaying asymptotically,

$$E \rightarrow E_0 \frac{c_-}{(-kx)^{d_-}}, \quad (56)$$

where $d_- > 3$. Note that the constants c_-, d_- specifying the asymptotics of the field profile are allowed to be different from those for the $x \rightarrow +\infty$ asymptotics. Dropping again the subscripts for brevity, the potential function is

$$f = -1 + \frac{c}{d-1} (-u)^{-(d-1)}, \quad (57)$$

and the Klein-Gordon equation reduces to

$$\left(\partial_u^2 - \frac{2c}{(d-1)} \frac{1}{k^2 (-u)^{(d-1)}}\right) \varphi = 0. \quad (58)$$

By matching with the WKB form in the overlap region near u_L^- , we find for the left Bessel region $u_A^- < u < u_L^-$

$$\varphi = b_3 \sqrt{-u} K\left(\frac{1}{d-3}, \frac{2}{d-3} \sqrt{\frac{2c}{d-1}} \frac{1}{k(-u)^{\frac{d-3}{2}}}\right), \quad (59)$$

where

$$b_3 = \frac{e^S}{\pi} \frac{F(s_+, c_+)}{\sqrt{d_- - 3}}. \quad (60)$$

Here, the tunnel exponent S , as in Eq. (11), arises from the integral in the WKB exponent Eq. (40).

2. Fields vanishing beyond a finite point

Finally, we consider fields approaching zero at $u \rightarrow u_-$ as

$$E \rightarrow E_0 c_- (u - u_-)^{n_-}, \quad (61)$$

and vanishing beyond that point for all $u < u_-$. The potential is given by

$$f = -1 + \frac{c}{n+1} (u - u_-)^{n+1}, \quad \text{for } u \gtrsim u_-, \quad (62)$$

and $f = -1$ for $u \leq u_-$. The Klein-Gordon equation reduces to

$$\left(\partial_{v_-}^2 - \frac{2c}{(n+1)k^2} v_-^{n+1}\right) \varphi = 0, \quad (63)$$

where $v_- = u - u_-$. A similar matching as above gives us the solution in the left Bessel region,

$$\varphi = b_3 \sqrt{v_-} K\left(\frac{1}{3+n}, \frac{2}{3+n} \sqrt{\frac{2c}{1+n}} \frac{v_-^{\frac{3+n}{2}}}{k}\right), \quad (64)$$

where

$$b_3 = \frac{e^S}{\pi} \frac{F(s_+, c_+)}{\sqrt{3+n_-}}. \quad (65)$$

For fields vanishing at either u_+ or u_- or both, it is obvious that the integrand in S given by Eq. (11) is identically zero beyond these points.

C. The left asymptotic region and the final result for D

We are now in a position to obtain D in Eq. (33). For fields decaying asymptotically in the left Bessel region, we take the $u \rightarrow -\infty$ limit of Eq. (59) and compare with Eq. (30), and for fields vanishing at a finite point in the left Bessel region we take the $v \rightarrow 0$ limit of Eq. (64). In all cases we find

$$|D| = \frac{e^S}{2\pi} F(s_-, c_-) F(s_+, c_+), \quad (66)$$

where $s = n$ or $s = -d$ depending on the asymptotic field properties, and F as in Eq. (54). This is the final result valid for weak fields $E_0 \ll 1$ and for all n and $d > 3$.

To check Eq. (66), we solve Eq. (14) numerically for the electric field shown in Fig. 8. The ratio of the amplitude squared obtained analytically and numerically is shown in Fig. 9. (See Fig. 10 for the corresponding plot in the strong field regime.) For some fields, a quantitative comparison requires to choose E_0 rather small depending on the desired precision of D , because the regions where it is sufficient to keep the first order corrections to $f = \pm 1$ have to partly overlap with the WKB region; due to the

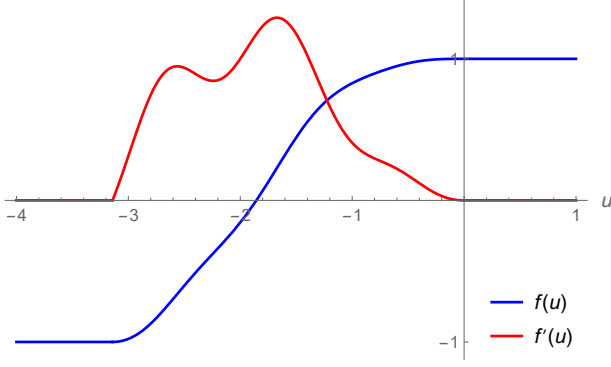


FIG. 8. Compact field example, defined by (126). The blue curve is the potential $f(kx)$ and the red curve is the electric field $E(kx)/E_0 = f'$. The electric field is identically zero for $kx > 0$ and $kx < -\pi$.

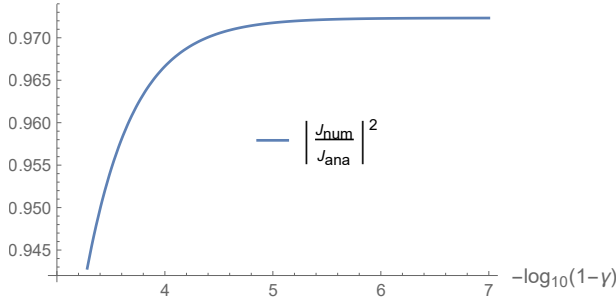


FIG. 9. Ratio of the squared amplitude $|J|^2$ obtained numerically and analytically. The field shape is that shown in Fig. 8, the field strength is $E_0 = 0.07$, and the momentum parameters are $r = 0$ and $v = 1$. The plot shows that the analytical approximation becomes better as $\gamma \rightarrow 1$.

exponential suppression, this can make the corresponding $\text{Im } \Gamma$ very small. We emphasize, though, that this does not affect the scaling.

As a further check, we consider the limit $d \rightarrow \infty$. From our previous paper, we expect this to give the same result as an exponentially decaying field, in particular as the

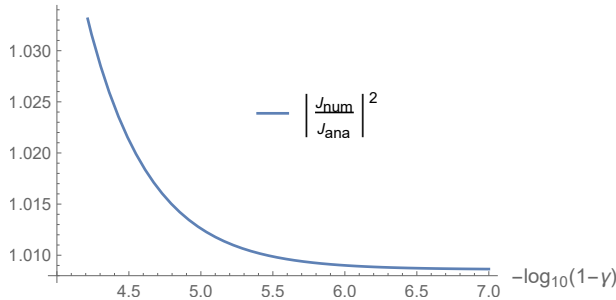


FIG. 10. Ratio of $|J|^2$ obtained numerically and analytically using Eq. (37). The field shape is that shown in Fig. 8, the field strength is $E_0 = 10$, and the momentum parameters are $r = 0$ and $v = 1$. The plot shows that the analytical approximation becomes better as $\gamma \rightarrow 1$.

exact result for the Sauter field [39]. Writing

$$e^{-\kappa u} = \lim_{d \rightarrow \infty} \left(1 + \frac{\kappa u}{d}\right)^{-d} \quad (67)$$

and casually exchanging the order of the limits $d \rightarrow \infty$ and $u \rightarrow \infty$, suggests that we should scale c in Eq. (42) as $c = (d/\kappa)^d \bar{c}$ with some d -independent constant \bar{c} . With this rescaling, the $d \rightarrow \infty$ limit of Eq. (66) is $D = \kappa e^S/2\pi$, and the imaginary part of the effective action becomes

$$\text{Im } \Gamma = \frac{\pi}{12} L^2 T \frac{(1 - \gamma^2)^3}{\kappa^2 E^2} e^{-2S}. \quad (68)$$

For the Sauter field, we have $\kappa = 2$ and $S = \pi/E$ from Eq. (11), which indeed agrees with the exact result [39].

We also recover this result from a limit of compact fields. In order to compare with an exponentially decaying field $E \rightarrow E_0 \bar{c} e^{-\kappa u}$ we choose $u_{\pm} = \pm n/\kappa$ and $c = \bar{c}(\kappa/n)^n$, so

$$c(u_+ - u)^n = \bar{c} \left(1 - \frac{\kappa u}{n}\right)^n \quad (69)$$

and similarly for u_- . In the limit $n \rightarrow \infty$ we again find $D = \kappa e^S/2\pi$ and thus Eq. (68).

D. Exponentially decaying fields

We have just shown that the result for exponentially decaying fields can be obtained from limits of fields vanishing with a power either asymptotically or beyond finite points. For completeness, we derive the same results directly in this subsection by starting with fields with asymptotic behavior

$$E \rightarrow E_0 c \kappa e^{-\kappa|u|}. \quad (70)$$

Here, c and κ may be chosen differently for $x \rightarrow \pm\infty$ in order to allow for fields vanishing non-symmetrically. For $u \gg 1$ we have

$$f = 1 - c e^{-\kappa u}. \quad (71)$$

For such fields, we can solve the Klein-Gordon equation in the Bessel, the semi-asymptotic and the asymptotic regions in one fell swoop. The solution to

$$\left(\partial_u^2 + \frac{1}{k^2} \left[-2c e^{-\kappa u} + q^2\right]\right) \varphi = 0 \quad (72)$$

can be written as

$$\varphi = \Gamma\left(1 - \frac{2iq}{\kappa k}\right) \left(\frac{2c}{\kappa^2 k^2}\right)^{\frac{iq}{\kappa k}} I\left(-\frac{2iq}{\kappa k}, \frac{2\sqrt{2c}}{\kappa k} e^{-\frac{\kappa u}{2}}\right). \quad (73)$$

By matching with the WKB form (40), we find

$$A = \sqrt{\frac{\kappa k}{4\pi}}. \quad (74)$$

This agrees with the $n \rightarrow \infty$ and $d \rightarrow \infty$ limits of Eq. (55). By a similar matching in the left Bessel region we find

$$D = \frac{\sqrt{\kappa_- \kappa_+} e^S}{2\pi}, \quad (75)$$

which agrees with Eq. (68) in the simplifying limit $\kappa_- = \kappa_+$.

VI. WEAK FIELDS DECAYING AS $E \sim x^{-3}$

All the fields considered so far, both compact and asymptotically decaying with power $d > 3$, have the same universal power-law scaling with critical exponent $\beta = 3$. In this section, we consider weak fields $E_0 \ll 1$ for the special case $d = 3$, and for simplicity we assume a symmetric decay

$$E \rightarrow E_0 \frac{c}{|kx|^3} \quad x \rightarrow \pm\infty. \quad (76)$$

In our previous paper [7], we found that these fields have power-law scaling with a critical exponent depending on the field strength in the semiclassical regime. We show here that the same scaling is recovered also in the deeply critical regime.

For $u \gg 1$ we have

$$f = 1 - \frac{c}{2u^2}. \quad (77)$$

Because the integrand in S goes like $1/u$ we cannot choose the upper integration limit as in Eq. (40). Instead we take

$$\varphi = \frac{A}{(1-f^2)^{\frac{1}{4}}} \exp \int_u^U \frac{\sqrt{1-f^2}}{k}, \quad (78)$$

with U large but finite. The limit $U \rightarrow \infty$ is considered on the level of the final result for $\text{Im } \Gamma$. The solution to

$$\left(\partial_u^2 + \frac{1}{k^2} \left[-\frac{c}{u^2} + q^2 \right] \right) \varphi = 0 \quad (79)$$

is given in terms of a Hankel function (Bessel function of the third kind) [56]

$$\varphi = \sqrt{\frac{\pi q u}{2k}} e^{\frac{i\pi}{4}(1+\frac{2\sqrt{c}}{k})} H^{(1)}\left(\frac{\sqrt{c}}{k}, \frac{qu}{k}\right), \quad (80)$$

where the normalization coefficient is chosen such that $\varphi \rightarrow e^{iqx}$ asymptotically. We have neglected some factors that are small due to $k \ll 1$, as we are working in the critical regime $\gamma = k/E_0 \rightarrow 1$ and at weak fields $E_0 \ll 1$. In the overlap with the WKB region, we have

$$\varphi = A c^{-\frac{1}{4}} u^{\frac{1}{2} - \frac{\sqrt{c}}{k}} e^{\frac{\sqrt{c}}{k} \ln U}, \quad (81)$$

which, upon comparing with Eq. (80), gives us A . In the overlap between the WKB and the left Bessel region, the solution is

$$\varphi = A c^{-\frac{1}{4}} (-u)^{\frac{1}{2} + \frac{\sqrt{c}}{k}} e^{S_\Lambda - \frac{\sqrt{c}}{k} \ln U}, \quad (82)$$

where

$$S_U = \int_{-U}^U du \frac{\sqrt{1-f^2}}{k}. \quad (83)$$

It follows that in the left Bessel region we have

$$\varphi \propto \sqrt{-u} J\left(\frac{\sqrt{c}}{k}, -\frac{pu}{k}\right), \quad (84)$$

with asymptotic limit

$$\varphi \propto \cos\left(px + \frac{\pi}{4}\left[1 + \frac{2\sqrt{c}}{k}\right]\right) \quad u \rightarrow -\infty. \quad (85)$$

By matching the different forms of the solution we find

$$|J| = \frac{1}{2\pi} \sqrt{\frac{q}{p}} \frac{\sqrt{c}}{k} \Gamma^2\left(\frac{\sqrt{c}}{k}\right) \left(\frac{4k^2}{pq}\right)^{\frac{\sqrt{c}}{k}} e^{S_U - \frac{2\sqrt{c}}{k} \ln U}. \quad (86)$$

Thus the imaginary part of the effective action scales as

$$\text{Im } \Gamma \sim (1 - \gamma^2)^{2(1+\sqrt{c}/k)}, \quad (87)$$

which we recognize from our semiclassical results in [7]. Thus, in contrast to the fields considered in the previous sections with n and $d > 3$, weak fields decaying with $|x|^{-3}$ have the same scaling in both the semiclassical and the deeply critical regime.

Although S_U diverges as $U \rightarrow \infty$, the limit of Eq. (86) is finite. Consider for example a Lorentz type of field,

$$f = \frac{u}{\sqrt{1+u^2}}, \quad (88)$$

for which $c = 1$. Expanding in U leads to

$$S_U - \frac{2}{k} \ln U = \frac{\ln 4}{k} + \mathcal{O}(U^{-2}). \quad (89)$$

It follows from this, together with Eq. (86) and Eq. (23), that

$$\text{Im } \Gamma = \frac{L^2 T \sqrt{\pi}}{16(2\pi)^2} E^{\frac{3}{2}} (1 - \gamma^2)^{2(1+\frac{1}{E})} \left[\frac{e}{4}\right]^{\frac{4}{E}}, \quad (90)$$

which, again, equals the critical limit of the semiclassical result in [37]. In general, the exponent in Eq. (86) can be expressed as

$$S_U - \frac{2\sqrt{c}}{k} \ln U = \frac{1}{k} \int_{-U}^U du \sqrt{1-f^2} - \theta(|u|-1) \frac{\sqrt{c}}{|u|}, \quad (91)$$

which makes it clear that the limit $U \rightarrow \infty$ is finite.

VII. FIELDS DECAYING AS $E \sim |x|^{-2}$

In this section we will consider fields decaying as

$$E \rightarrow E_0 \frac{c}{|kx|^2}. \quad (92)$$

As for weak fields with $d = 3$, we also recover the semiclassical scaling as shown in the following. For $u \gg 1$ we have

$$f = 1 - \frac{c}{u}, \quad (93)$$

and the Klein-Gordon equation reduces to

$$\left(\partial_u^2 + \frac{1}{k^2} \left[-\frac{2c}{u} + q^2\right]\right)\varphi = 0. \quad (94)$$

Instead of a Bessel function, this time the solution is given by a Whittaker function¹

$$\varphi = \left(-\frac{2iq}{k}\right)^{\frac{ic}{kq}} W_{-\frac{ic}{kq}, \frac{1}{2}} \left[-\frac{2iqu}{k}\right]. \quad (95)$$

The asymptotic limit $u \rightarrow \infty$ of Eq. (95) is not simply e^{iqx} , but

$$\varphi \rightarrow \exp i \left(qx - \frac{c}{kq} \ln(qx) + \text{real constant} \right). \quad (96)$$

However, we still have the same normalization

$$\varphi^\dagger (-i \overleftrightarrow{\partial}) \varphi \rightarrow 2q, \quad (97)$$

so we expect that the relation between \mathcal{T} and $\text{Im } \Gamma$ derived in the literature is still valid.

A. Weak fields

We begin with the scaling for weak fields $E_0 \ll 1$. In the overlap with the WKB regime $u \sim u_L^+$, where

$$1 \ll u \ll \frac{c}{k^2} \ll \frac{c}{q^2}, \quad (98)$$

we use the asymptotic expansions in [61] for $W_{\kappa, \frac{1}{2}}(z)$ as $|\kappa| \rightarrow \infty$, yielding

$$\varphi = \sqrt{q} \left(\frac{u}{2c}\right)^{\frac{1}{4}} \exp \left(\frac{\pi c}{kq} - \frac{2\sqrt{2cu}}{k} + i \dots \right). \quad (99)$$

By matching this with the WKB form (78), we find

$$|A| = \sqrt{q} \exp \left(\frac{\pi c}{kq} - \frac{2\sqrt{2cU}}{k} \right). \quad (100)$$

Already at this point, we can see the emergence of essential scaling by recalling $q \sim \sqrt{1 - \gamma^2}$. To the left of the WKB region the solution is given by another Whittaker function

$$\varphi = b_2 M_{\pm \frac{ic}{kp}, \frac{1}{2}} \left[\pm \frac{2ip(-u)}{k} \right]. \quad (101)$$

The asymptotic limit $u \rightarrow -\infty$ has the form of Eq. (17) modified with logarithmic terms as in Eq. (96). Using again expansions given in [61], we find

$$|J| = \sqrt{\frac{q}{p}} \exp \left[\frac{\pi c}{k} \left(\frac{1}{p} + \frac{1}{q} \right) + \left(\int_{-U}^U \frac{\sqrt{1 - f^2}}{k} \right) - \frac{4\sqrt{2cU}}{k} \right]. \quad (102)$$

Substituting this into Eq. (23) and taking $U \rightarrow \infty$, we finally obtain

$$\begin{aligned} \text{Im } \Gamma &= \frac{L^2 T}{32\sqrt{6}\pi^3} \left(\frac{k}{c}\right)^{\frac{3}{2}} (1 - \gamma^2)^{\frac{1}{4}} \exp \left(-\frac{4\pi c}{k\sqrt{1 - \gamma^2}} \right) \\ &\times \exp \left\{ -\frac{2}{k} \left[\left(\int_{-U}^U \sqrt{1 - f^2} \right) - 4\sqrt{2cU} \right] \right\} \Big|_{U \rightarrow \infty}, \end{aligned} \quad (103)$$

where the second line can asymptotically be expressed as

$$\exp \left[-\frac{2}{k} \int_{-\infty}^{\infty} du \left(\sqrt{1 - f^2} - \sqrt{\frac{2c}{|u|}} \right) \right]. \quad (104)$$

This is indeed exactly the same scaling as we found in the semiclassical regime [7], which suggests that our semiclassical results in [7] hold more generally in the deeply critical regime for weak fields with $2 \leq d \leq 3$. For the field implicitly defined by $f' = (1 - f^2)^2$ the term in square brackets in the second line in Eq. (103) vanishes and one can show that the prefactor in the first line of Eq. (103) also agrees with the semiclassical result.

B. Strong fields

Now let us study strong fields $E_0 \gg 1$. We consider again for simplicity symmetric fields. Let U be such that for $u > U$ we have $f = 1 - c/u$. Although $U \gg 1$ we still have $qU/k \ll 1$, so in the region $u \sim U$ we can expand Eq. (95)

$$\varphi = Q_0 \left(1 + \frac{2cu}{k^2} \left[\ln \left(\frac{2cu}{k^2} e^{2\gamma_E} \right) - 1 \right] \right), \quad (105)$$

where $\gamma_E \approx 0.577$ is the Euler number and

$$|Q_0| = \sqrt{\frac{kq}{2\pi c}} e^{\frac{\pi c}{kq}}. \quad (106)$$

In the region $-U < u < U$ we solve the Klein-Gordon equation (34) perturbatively in $1/k$,

$$\varphi = Q_0 \left(1 + \frac{2c}{k^2} \left[c_0 + c_1 u + \int_U^u du' (u - u') \frac{1 - f^2(u')}{2c} \right] \right), \quad (107)$$

¹ Whittaker functions also appear in treatments of pair production by constant electric fields in de Sitter space, see e.g. [57–60] and references therein.

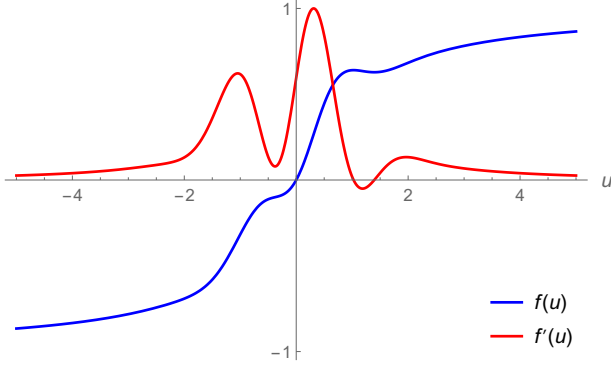


FIG. 11. Example field defined by (127) that vanishes as $E \sim |x|^{-2}$; the blue curve is the potential $f(u)$ and the red curve the normalized electric field $E/E_0 = f'$.

where the two constants c_0 and c_1 are obtained by matching Eq. (107) with Eq. (105) in the region $u \sim U$. For $u < -U$ the solution is given by a linear combination of Whittaker functions,

$$\varphi = AM_{\frac{ic}{kp}, \frac{1}{2}} \left[-\frac{2ipu}{k} \right] + BW_{\frac{ic}{kp}, \frac{1}{2}} \left[-\frac{2ipu}{k} \right]. \quad (108)$$

By matching Eq. (108) with Eq. (107), we find $B = \Gamma(1 - ic/kp)Q_0$ and

$$A = -\frac{icQ_0}{kp} \left[\left(\int_{-U}^U \frac{1-f^2}{2c} \right) - 2 \ln \frac{2cUe^{2\gamma_E}}{k^2} \right], \quad (109)$$

where the limit $U \rightarrow \infty$ of the term in square brackets is finite and given by

$$\int_{-\infty}^{\infty} du \left(\frac{1-f^2}{2c} - \frac{\theta(|u|-1)}{|u|} \right) - 2 \ln \frac{2ce^{2\gamma_E}}{k^2}. \quad (110)$$

In the asymptotic limit $u \rightarrow -\infty$ the first term in Eq. (108) is dominant and the amplitude becomes

$$|J| = \frac{1}{2\pi} \sqrt{\frac{q}{p}} \left| \left(\int_{-U}^U \frac{1-f^2}{2c} \right) - 2 \ln \frac{2cUe^{2\gamma_E}}{k^2} \right| \times \exp \frac{\pi c}{k} \left(\frac{1}{p} + \frac{1}{q} \right). \quad (111)$$

To check these analytical expressions, we consider the field depicted in Fig. 11, and show in Fig. 12 that the analytical approximation for $|\varphi(u)|$ agrees well with the exact numerical solution.

By comparing Eq. (111) with Eq. (102) we see that the critical scaling for strong fields is the same as that for weak fields. The prefactor, though, is different. In fact, for very strong fields we can drop the integral term in Eq. (111),

$$|J| \approx \frac{2}{\pi} \sqrt{\frac{q}{p}} \ln k \exp \frac{\pi c}{k} \left(\frac{1}{p} + \frac{1}{q} \right). \quad (112)$$

Thus, for very strong fields vanishing as $E \sim |x|^{-2}$, the whole expression for $\text{Im } \Gamma$ is universal and not just the scaling behavior (c.f. Eq. (13)).

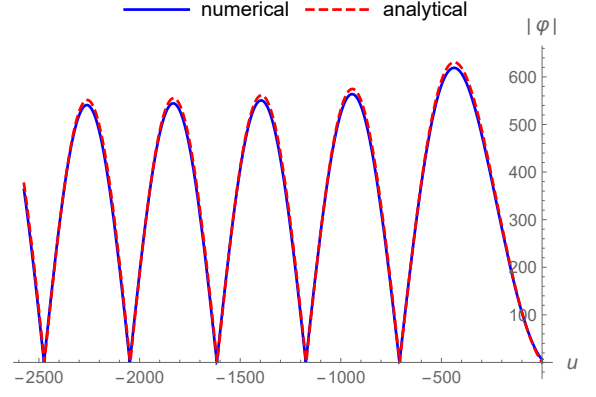


FIG. 12. Modulus of the solution to the Klein-Gordon equation $|\varphi(u)|$ in the left asymptotic region, i.e., for large negative u . The field shape is that shown in Fig. 11 with field strength $E_0 = 10$, adiabaticity $1 - \gamma = 3 * 10^{-3}$, and momentum parameters $r = 0$ and $v = 1$. The two curves show that the analytical results (107), (109) and (110) agree well with the exact numerical solution; the analytical solution converges to the exact numerical one as $1 - \gamma$ decreases.

VIII. SPINOR QED

So far we have considered scalar QED. In this section, we briefly consider spinor QED [39]. The imaginary part of the effective action is still given by Eq. (19), but the tunneling factor now reads

$$\mathcal{T} = 2 \frac{p_0 + p}{2/\gamma - p_0 - q} \frac{q}{p} \frac{1}{|J|^2}, \quad (113)$$

where the factor of 2 comes from the sum over spin degrees of freedom. In the critical regime, this reduces to

$$\mathcal{T} = 2 \frac{q}{p} \frac{1}{|J|^2}, \quad (114)$$

which has the same form as the scalar case in Eq. (18). The amplitude J is obtained from φ as in Eq. (17), but φ is now the solution of the squared Dirac equation,

$$\left(\partial_x^2 + [p_0 - A(x)]^2 - m_\perp^2 + iA'(x) \right) \varphi = 0, \quad (115)$$

which in the critical regime reduces to

$$\left(\partial_u^2 + \frac{1}{k^2} \left[-(1-f^2) + ikf' + P^2 \right] \right) \varphi = 0, \quad (116)$$

where $P^2 = q^2$ for $u \gtrsim 0$, and $P^2 = p^2$ for $u \lesssim 0$. Comparing with the scalar case we see that the f' term is new. It arises from the Pauli term $\sim \sigma_{\mu\nu} F^{\mu\nu}$ of the squared Dirac operator. For weak fields the solution in the WKB region is

$$\varphi = \frac{A}{(1-f^2)^{\frac{1}{4}}} \exp \left(\frac{i}{2} \sin^{-1} f + \int_u^\infty \frac{\sqrt{1-f^2}}{k} \right). \quad (117)$$

For asymptotically decaying weak fields the new f' term is always smaller than the $1 - f^2$ term: in the Bessel region we have

$$kf' \sim \frac{k}{u^d} \ll \frac{1}{u^d} \ll \frac{1}{u^{d-1}} \sim 1 - f^2, \quad (118)$$

and in the overlap with the WKB region the extra term in Eq. (117) simply gives an irrelevant phase. Thus, scalar and spinor QED agree in the deeply critical regime (up to a factor of 2) for asymptotically decaying weak fields. This is also reflected by the exact result for a Sauter pulse [39], which for spinor QED leads to Eq. (68) times a factor of 2. For compact fields we have in the Bessel region

$$\frac{kf'}{1 - f^2} \sim \frac{k}{u_+ - u}, \quad (119)$$

so the spin term actually becomes larger than the scalar term as $u \rightarrow u_+$, which suggests that the spin term can lead to corrections to the non-universal coefficient D . However, the spin term is only dominant in a region of size $u_+ - u \lesssim k$, so these corrections will not change D significantly. Indeed, the most important part of the D is still given by e^S as in the scalar case. Moreover, the critical exponent remains unchanged,

$$\text{Im } \Gamma_{\text{spin}} \sim (1 - \gamma^2)^3. \quad (120)$$

For strong fields, a remarkable simplification occurs in the spinor case marking a qualitative difference to the scalar case: we can find D_{spin} by going back to Eq. (37) and replacing $1 - f^2$ with $1 - f^2 - ikf'$. While $1 - f^2$ is bounded, the term $\sim kf' \sim E$ is proportional to the electric field, thus dominating the integrand in the strong-field limit,

$$\begin{aligned} D_{\text{spin}} &= \int_{-\infty}^{\infty} du \frac{1 - f^2(u) - ikf'(u)}{k^2} \\ &\approx -\frac{i}{k} \int f' = -\frac{2i}{k}, \end{aligned} \quad (121)$$

and hence we find Eq. (13). Equation (13) agrees with the exact result for a Sauter pulse [39] in the strong-field limit. In Figs. 14 and 16, we verify that Eq. (121) matches with the numerical solution of Eq. (115) near the critical point for the fields shown in Figs. 13 and 15, respectively.

The origin of this enhanced strong-field universality lies in the dominance of the Pauli term $\sim \sigma_{\mu\nu} F^{\mu\nu}$, parametrizing the coupling of the spin structure to the field. It is instructive to recall the relevance of this term in the analog case of a strong magnetic field: in the magnetic case, the Pauli term characterizes the paramagnetism of the electron-positron fluctuations as opposed to the Klein-Gordon operator describing diamagnetism. In the strong-field limit, paramagnetism dominates which is reflected by a zero-mode of the squared

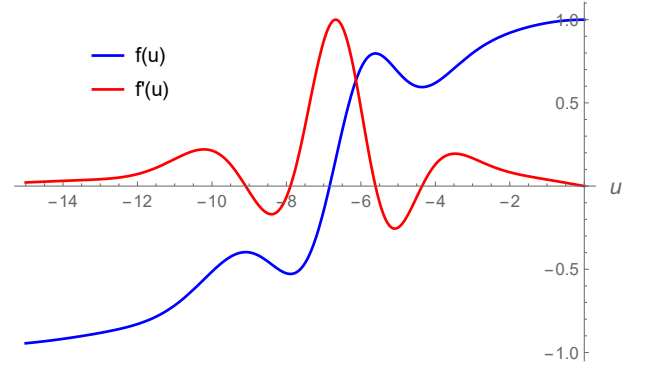


FIG. 13. Example field as defined in Eq. (128). The blue curve depicts the potential function $f(kx)$ and the red curve is the electric field $E(kx)/E_0 = f'$.

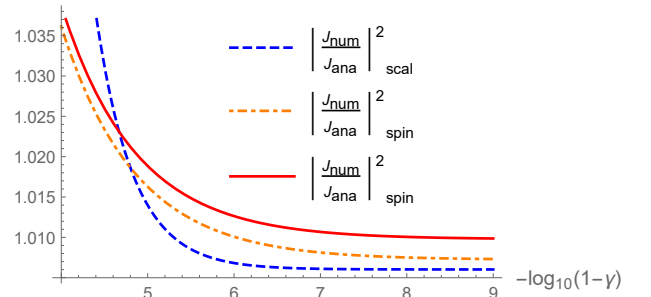


FIG. 14. Ratios of the squared amplitude obtained numerically and analytically for scalar and spinor QED. The field is shown in 13 with peak-field parameter $E_0 = 70$, and the momentum parameters are $r = 0$ and $v = 1$. For scalar QED, J_{num} is obtained from the numerical solution of Eq. (14), J_{ana} is the analytical estimate of Eq. (37), and their ratio approaches $\approx 1 - 6 \cdot 10^{-3}$. For spinor QED, J_{num} is obtained from the numerical solution of Eq. (115). The latter is compared to two different analytical estimates J_{ana} deduced from Eq. (121) and Eq. (31) by either keeping (orange dot-dashed curve) or neglecting (red solid curve) the $1 - f^2$ term, and their ratios approach $\approx 1 - 7 \cdot 10^{-3}$ and $\approx 1 - 9.9 \cdot 10^{-3}$, respectively.

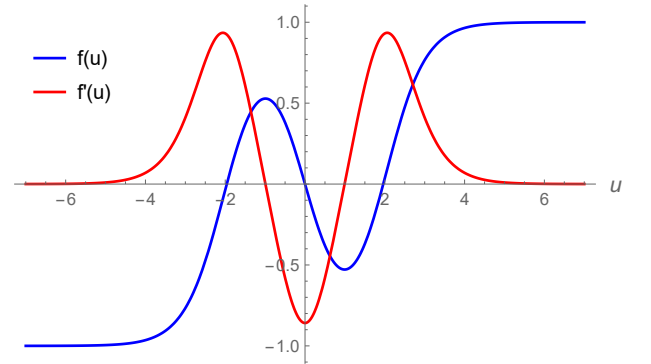


FIG. 15. Example field as defined in Eq. (129). The blue curve depicts the potential function $f(kx)$ and the red curve is the electric field $E(kx)/E_0 = f'$.

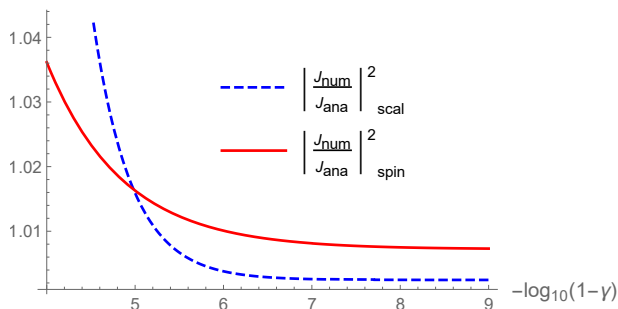


FIG. 16. Ratios of the squared amplitude obtained numerically and analytically for scalar and spinor QED, as described in Fig. 14. The field is shown in 15, the field strength is $E_0 = 70$ and the momentum parameters are $r = 0$ and $v = 1$. The scalar and spinor ratios converge to $\approx 1 - 2.5 \times 10^{-3}$ and $\approx 1 - 7.3 \times 10^{-3}$.

Dirac operator for the lowest Landau level and a suitably oriented spin [43, 62]. This mode is responsible for a variety of strong-field features that are unique to spinor QED [63–68]. In the present case, it is again the Pauli term dominating the strong field limit, even though the Minkowskian mode structure in the electric field is somewhat different from the magnetic case: e.g., there is no dependence on the orientation of the spin relative to the field, as the electron does not have a permanent electric dipole. Still, the analogy to the magnetic case justifies to classify the enhanced universality of deeply critical pair production in the spinor case as *paraelectric dominance*. This nomenclature reflects the dominance of the endomorphism $\sim \sigma_{\mu\nu} F^{\mu\nu}$ in the relevant differential operator as opposed to the “dia”-part of the covariant Laplacian [69].

IX. CONCLUSION

We have studied critical scaling in Schwinger pair production. In the space of all possible electromagnetic field configurations, we have considered the regime near the critical hypersurface that separates the configurations that allow for pair production from those that do not. This critical surface can be detected using the imaginary part of the Heisenberg-Euler effective action $\text{Im } \Gamma$ as an order parameter. As noted in our previous work [7], the scaling of $\text{Im } \Gamma$ in the vicinity of the critical surface exhibits scaling laws that are reminiscent to critical phenomena in statistical systems. In the present paper, we generalize these previous results in two decisive aspects: we provide for the first time scaling results in the deeply critical regime, i.e. in the immediate vicinity of the critical surface, whereas the results of [7] apply to the semiclassical critical region. Second, by directly studying the underlying field equations, we can address a wider class of field profiles, also including non-symmetric cases with non-monotonic gauge potentials. For simplicity, we still restrict ourselves to uniaxial time-independent field pro-

files varying in one spatial direction.

In comparison with the semiclassical critical regime, the deeply critical regime supports an even higher degree of universality, with the same scaling law as a function of the Keldysh parameter γ ,

$$\text{Im } \Gamma \sim (1 - \gamma^2)^\beta, \quad \beta = 3, \quad (122)$$

for all fields that asymptotically vanish faster than $|x|^{-3}$. The result holds for both scalar and spinor QED in the weak- as well as strong-field regime. The existence of such a scaling law expresses the fact that the onset of pair production is dominated by the long-range fluctuations of the electron-positron field and becomes insensitive to the microscopic details of the field profile.

The highest degree of universality is found for spinor QED in the strong-field regime, where paraelectric dominance also establishes a universal prefactor. The scaling law Eq. (122) is modified for more gradually vanishing fields. E.g., fields that vanish as $|x|^{-2}$ obey the same scaling law as in the semiclassical regime. The scaling for fields vanishing as $|x|^{-3}$ depends on the field strength; for weak fields, we recover the semiclassical scaling, whereas Eq. (122) holds again for strong fields.

As we noted in [7], it is easy to generalize the results for the spatially inhomogeneous fields considered here to fields that depend on a linear combination of x and t as in [70]. It would be interesting to investigate how more nontrivial combinations of spatial and temporal inhomogeneities affect the critical scaling found in this paper. Of particular relevance for discovery experiments may be fields that support dynamically assisted pair production [15], e.g., by adding a weak time-dependent field to the fields considered here, e.g. as in [16].

Since the real and imaginary parts of correlation functions and thus of the effective action are related by dispersion relations, the onset of pair production in the critical regime will also leave an imprint in the real part of the effective action. One physical manifestation of such dispersion relations is, for instance, given by an anomalous dispersion of a photon propagating through the field, see e.g. [71], which can be extracted from the photonic two-point correlator [63, 72–75]. However, since the real part also receives contributions from perturbative processes, it remains an open question as to whether the analogue of the (nonperturbative) scaling behavior can be quantitatively dominant in the regime of anomalous dispersion.

Dispersion relations also lead to an intriguing relation between the imaginary part of the action and the large-order behavior of the perturbative expansion of the real part: for constant fields as well as for the spatially homogeneous electric Sauter profile in time, it has been shown that the absence/appearance of the imaginary part is tightly linked to the properties of the perturbative series under Borel transformations [48, 49, 76, 77] (see also [71]). If this pattern also holds for spatially inhomogeneous fields, the critical surface of Schwinger pair production in the space of field profiles could also separate Borel- from non-Borel-summable perturbative ex-

pansions of the effective action. Such a conjecture clearly deserves further investigation.

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APPENDIX

Here we list the field examples used above to demonstrate various results. The first field example, shown in Fig. 1, is given by $f(u) = f_0(u/\hat{f}'_0)$, where $f(u > 0) = 1$,

$$f_0(u < 0) = 1 + \left(1 + \frac{1}{5} \sin(13u) \operatorname{sech}^2 \left[3u + \frac{21}{10}\right]\right) \frac{4}{3\pi} \left[\frac{u(-3 + 8u^2 + 3u^4)}{(1 + u^2)^3} + 3 \arctan u \right] \quad (123)$$

and where $\hat{f}'_0 \approx 4.5$ is the maximum of f'_0 . The second field example, shown in Fig. 5, is given by

$$f(u) = \tan\left(\frac{u}{3}\right) - \frac{1}{5} \cos\left(\frac{10u}{3}\right) \exp\left(-\left(\frac{u}{3}\right)^2\right). \quad (124)$$

The compact example in Fig. 9 is obtained by first integrating

$$f'_0(u) = -u \cos\left(u + \frac{\pi}{2}\right) \left(1 + \frac{1}{4} \cos\left[5\left(u + \frac{\pi}{2}\right)\right]\right) \quad (125)$$

and then

$$f(-\pi < u < 0) = 1 - 2 \frac{f_0(0) - f_0(u)}{f_0(0) - f_0(-\pi)}, \quad (126)$$

$f(u < -\pi) = -1$ and $f(u > 0) = 1$. The field example Fig. 12 is given by $f(u) = f_0(u/\hat{f}'_0)$, where

$$f_0(u) = \frac{2}{\pi} \arctan\left[\frac{\pi}{2}u\right] \left(1 + \sin(3u) \exp(-3u^2)\right) \quad (127)$$

and where $\hat{f}'_0 \approx 1.7$ is the maximum of f'_0 . An example similar to (123), but with simpler analytical form, is given by $f(u) = f_0(u/\hat{f}'_0)$, where $f_0(u > 0) = 1$,

$$f_0(u < 0) = 1 - \frac{2u^2}{(1 + u^6)^{\frac{1}{3}}} \left(1 + \frac{7}{10} \sin[14u] \operatorname{sech}^2[5u + 3]\right) \quad (128)$$

and where $\hat{f}'_0 \approx 10$ is the maximum of f'_0 . Eq. (128) is shown in Fig. 13. The symmetric field shown in Fig. 15 is defined by

$$f(u) = \tanh(u + 2) - \tanh(u) + \tanh(u - 2). \quad (129)$$

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